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Search for Duality Symmetries in p-Branes *

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Abstract

The requirement of an $SL(2)$ duality symmetry, mixing the worldvolume field equations with Bianchi identities, leads to a highly nonlinear equation involving the transformation parameters and certain worldvolume currents. In general, this equation seems to admit a solution only for a two parameter subgroup of the seeked $SL(2)$. These transformations also leave invariant the first class constraints generating the worldvolume reparametrizations. In the special case of p–branes in $p+1$ dimensions, the full $SL(2)$ is realized.

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I met Feza Gürsey a number of times, but unfortunately I never had a chance to collaborate with him. I have always been inspired by his beautiful work and his kind and gentle personality. I have a great respect and admiration for him. He is greatly missed and he will always be remembered.

Feza Gürsey had a keen sense for beauty of symmetries in physics and he made profound contributions in search of them. I believe that the topic of this talk, which is dedicated to his memory, is very much in the spirit of his research philosophy that emphasizes symmetries. More specifically, I will talk about a search for duality symmetries in theories of extended objects, known as p-branes.

It is well known that the ten dimensional heterotic string theory compactified on a six torus has target space duality symmetry group $O(6,22;Z)$ which mixes momentum modes with winding modes [1,2]. From the world-sheet field theory point of view, this group mixes the world-sheet field equations with Bianchi identities [2]. It is natural to search for similar duality symmetries for higher p–branes.

In view of the possibility of a connection beetween the strongly coupled heterotic string and a weakly coupled fivebrane theory in ten dimensions [3], the issue of duality symmetry in fivebrane theory especially seems interesting. The expected target space duality group in this case is an $SL(2,Z)$ group [4] (For a review, see [5]). Although it has been shown that an $SL(2,Z)$ duality group indeed mixes the momentum modes with the winding modes of the fivebrane theory [6], so far this symmetry group has not been understood as a worldvolume duality group transforming the worldvolume equations of motion into Bianchi identities. In a recent paper [7], we studied the problem from this point of view. We found that, with a reasonable set of assumptions about the form of the duality transformations, the existence of an $SL(2)$ symmetry requries a solution to a highly nonlinear equation involving the transformation parameters and the worldvolume currents that transform into each other under duality transformations. In general, this equation seems to admits only a two parameter solution which forms a subgroup of the seeked $SL(2)$. In the special case of p–branes in $p+1$ dimensions, the full $SL(2)$ is realized. In this note, we will describe the main result of [7], and we shall furthermore show that the two parameter duality group also leaves invariant the first class constraints which generate the worldvolume reparametrizations. An attempt to find duality symmetries in p–branes has been made before in somewhat different setting [8]. Further comments on this paper can be found in [7].

While the case of most interest is the fivebrane, we shall consider all p–branes, without much more effort. The dynamical variables describing the p–brane are scalar fields $x^\mu(\sigma)$, $y^\alpha(\sigma)$ and a worldvolume metric $\gamma_{ij}(\sigma)$. Here σ^i $(i=0,...,p)$ are the worldvol-

ume coordinates, x^μ, $\mu=0,...,m-1$ are coordinates on m dimensional space M and y^α, $\alpha=1,...,p+1$ are coordinates on a compact $(p+1)$-dimensional manifold N. We shall take the background fields to be the metrics $g_{\mu\nu}(x)$ and $g_{\alpha\beta}(x)$ on M and N respectively and an antisymmetric tensor field $b_{\alpha_1\ldots\alpha_{p+1}}(x)=\lambda_1(x)\epsilon_{\alpha_1\ldots\alpha_{p+1}}$. In this background, the usual Polyakov type action for the p–brane is

$$S=\int d^{p+1}\sigma\mathcal{L}=\int d^{p+1}\sigma\Big[-\frac{1}{2}\sqrt{-\gamma}\left(\gamma^{ij}\partial_i x^\mu\partial_j x^\nu g_{\mu\nu}+\gamma^{ij}\partial_i y^\alpha\partial_j y^\beta g_{\alpha\beta}\right)+\frac{p-1}{2}\sqrt{-\gamma}$$
$$+\frac{1}{(p+1)!}\epsilon^{i_1\ldots i_{p+1}}\partial_{i_1}y^{\alpha_1}\cdots\partial_{i_{p+1}}y^{\alpha_{p+1}}\lambda_1\epsilon_{\alpha_1\ldots\alpha_{p+1}}\Big]\ . \tag{1}$$

Let us concentrate on the field equation for the internal coordinates y^α. It reads $\partial_i P^i{}_\alpha=0$, where

$$P^i{}_\alpha=\frac{\partial\mathcal{L}}{\partial\partial_i y^\alpha}=-\sqrt{-\gamma}\gamma^{ij}\partial_j y^\beta g_{\beta\alpha}+\lambda_1 J^i{}_\alpha\ , \tag{2}$$

$$J^i{}_\alpha=\frac{1}{p!}\epsilon^{ij_1\ldots j_p}\partial_{j_1}y^{\beta_1}\cdots\partial_{j_p}y^{\beta_p}\epsilon_{\alpha\beta_1\ldots\beta_p}\ . \tag{3}$$

In searching for a duality symmetry in p–brane theories, we also need to know how the induced metric on the worldvolume transforms. We know from ten dimensional supergravity compactified on a six–torus that under $SL(2)$ the metrics $g_{\alpha\beta}$ and $g_{\mu\nu}$ rescale. Therefore let us define

$$g_{\mu\nu}=\lambda_2^K\bar{g}_{\mu\nu}\ ,\qquad\qquad g_{\alpha\beta}=\lambda_2^L\bar{g}_{\alpha\beta}\ , \tag{4}$$

where $\bar{g}_{\alpha\beta}$ and $\bar{g}_{\mu\nu}$ are assumed to be inert under $SL(2)$, and det $\bar{g}_{\alpha\beta}=1$. Thus $\lambda_2(x)=(\det g_{\alpha\beta})^{1/(p+1)L}$. In the case $p=5$ it is known from the $SL(2)$ duality symmetry of the effective field theory limit that $K=-1$ and $L=1/3$ [5].

The equation of motion for the worldvolume metric γ gives

$$\gamma_{ij}=\lambda_2^K\partial_i x^\mu\partial_j x^\nu\bar{g}_{\mu\nu}+\lambda_2^L\partial_i y^\alpha\partial_j y^\beta\bar{g}_{\alpha\beta}\ . \tag{5}$$

Thus, the duality transformation rule for the worldvolume metric follows from the transformation rules assigned to quantitities occuring in this equation.

We now look for transformation rules that mix the field equation $\partial_i P^i{}_\alpha=0$, with the Bianchi identity $\partial_i J^i{}_\alpha=0$. The most natural way to do this is to consider the transformations of the currents $P^i{}_\alpha$ and $J^i{}_\alpha$ into each other as

$$\delta P^i{}_\alpha=aP^i{}_\alpha+bJ^i{}_\alpha\ , \tag{6a}$$

$$\delta J^i{}_\alpha=cP^i{}_\alpha+dJ^i{}_\alpha\ , \tag{6b}$$

with a, b, c, d being constants. It is important to take the parameters to be constant, so that these transformations indeed map the field equations and Bianchi identities into a combination of each other. The key point in establishing the duality symmetry is to show that (2) is invariant under the transformations (6), combined with appropriate transformation rules for the background fields λ_1 and λ_2. In this regard, we are following the approach of Gaillard and Zumino [9].

Since all relevant quantities have two indices, it is convenient to use matrix notation. We define matrices P and J with components $P^i{}_\alpha$ and $J^i{}_\alpha$, matrices ∂y with components $(\partial y)^\alpha{}_i = \partial_i y^\alpha$ and $\bar{g}^{(p+1)}$ with components $\bar{g}_{\alpha\beta}$. From the definition (3) we find that $\partial y = J^{-1}(\det J)^{1/p}$. This equation allows us to calculate the variation of ∂y under the duality transformations.

The central result of [7] is that, taking into account the variation of the worlvolume metric γ, and allowing any transformation rules for the scalar fields λ_1, λ_2, the invariance of (2) under the duality transformations (6) implies requires that the following highly nonlinear equation be satisfied:

$$\begin{aligned} cX^2 \; + \; & \left[a - 2c\lambda_1 - \frac{1}{p}(d + c\mathrm{tr}X) - \left(\frac{p-1}{2}K + L\right)\lambda_2^{-1}\delta\lambda_2\right] X \\ & + b - \frac{p-1}{p}d\lambda_1 + \frac{1}{p}c\lambda_1\mathrm{tr}X + \left(\frac{p-1}{2}K + L\right)\lambda_1\lambda_2^{-1}\delta\lambda_2 - \delta\lambda_1 = \\ = & \lambda_2^L\gamma^{-1}V\Bigg\{2cX^2 - \left[\frac{2}{p}(d + c\mathrm{tr}X) + 2c\lambda_1 + (L-K)\,\lambda_2^{-1}\delta\lambda_2\right] X \\ & + \frac{1}{p}c(\mathrm{tr}X)^2 - c\mathrm{tr}(X^2) + \left(\frac{1}{p}c\lambda_1 + \frac{1}{p}d + \frac{1}{2}(L-K)\lambda_2^{-1}\delta\lambda_2\right)\mathrm{tr}X \\ & \qquad - \frac{p-1}{p}d\lambda_1 - \frac{1}{2}(p-1)(L-K)\lambda_1\lambda_2^{-1}\delta\lambda_2\Bigg\} , \end{aligned} \tag{7}$$

where

$$X = P\cdot J^{-1} = \lambda_1 + \lambda_2^L \frac{\gamma^{-1}V}{\sqrt{-\det(\gamma^{-1}V)}} , \qquad V = (\partial y)^T \bar{g}^{(p+1)}\partial y . \tag{8}$$

Since $\gamma^{-1}V$ can be reexpressed in terms of X via (8), this is an infinite polynomial equation in the $(p+1)\times(p+1)$ matrix X. One is free to determine $\delta\lambda_1$ and $\delta\lambda_2$ as functions of $a, b, c, d, \lambda_1, \lambda_2$ to satisfy this equation, and also if necessary to put restrictions on the transformation parameters a, b, c, d. It is important to realize that these transformations have to be the same for all X. In order to prove that this equation has no solution it would therefore be sufficient to find two particular matrices X which give incompatible values for the variations $\delta\lambda_1$ and $\delta\lambda_2$.

In [7], we implemented this idea by choosing a particular matrix X and expanding equation (7) around it. For our background we choose the fields x^μ and y^α such that $\gamma^{-1}V = \lambda_2^{-L}\eta$, where η is the Minkowski metric, and then write $\gamma^{-1}V = \lambda_2^{-L}(\eta + Y)$. Expanding (7) in powers of Y, we then find that the solution of (7) is given by

$$\begin{aligned} c =& 0 \ , \\ d =& p\frac{K-L}{pK+L}a \ , \\ \delta\lambda_1 =& b + \frac{(p+1)L}{pK+L}a\lambda_1 \ , \\ \delta\lambda_2 =& \frac{2}{pK+L}a\lambda_2 \ . \end{aligned} \tag{9}$$

In fact, one can check this is the solution of the full equation (7). Furthermore, one can show that the x^μ-equation of motion is also invariant under these transformations. Therefore, we have a two parameter group of duality transformations of the p–brane. It is easy to check that the transformation rules (6) and (9) yield the same commutator algebra. Denoting the transformations by a and b, the only nonvanishing commutator is $[a, b] = b$. One can have $d = -a$ by choosing $K = \frac{p-1}{2p}L$. In this way the two parameter group appears to be a subgroup of the expected group $SL(2)$. Notice that, from a solution with magnetic charge, one can obtain a solution with magnetic and electric charge. However, the two parameter group does not contain the important $R \to 1/R$ transformations.

In the special case of a p–brane propagating in $p + 1$ dimensions, it is straightforward to show that the full $SL(2)$ duality symmetry group is realized.

Finally, let us consider the action of the two parameter duality group described above on the reparametrization generating constraints. These constraints are [10]

$$\begin{aligned} H =& g^{\mu\nu}p_\mu p_\nu + g^{\alpha\beta}(p_\alpha - \lambda_1 j_\alpha)(p_\beta - \lambda_1 j_\beta) \\ & + \det\,(\partial_a x^\mu \partial_b x^\nu g_{\mu\nu} + \partial_a y^\alpha \partial_b y^\beta g_{\alpha\beta}) \ , \end{aligned} \tag{10}$$

$$H_a = \partial_a x^\mu p_\mu + \partial_a y^\alpha p_\alpha \ , \tag{11}$$

where the index $a = 1, ..., p$ labels the spatial directions, p_μ and p_α are the momentum variables associated with x^μ and y^α, respectively, and $j_\alpha = J^0{}_\alpha$, which can be read off from $J^i{}_\alpha$ given in (3).

Consistent with the definition (10) and the transformations rules (6) and (9) one finds $\delta\partial_a y^\alpha = \frac{d}{p}\partial_a y^\alpha$. Using this variation, and taking x^μ, and therefore $\partial_a x^\mu$ to be inert, we find that under the duality transformations (6) and (9) the constraint H_a is preserved: $\delta H_a = \frac{(p+1)K}{pK+L}aH_a$, provided that we assign the transformation rule $\delta p_\mu = \frac{(p+1)K}{pK+L}ap_\mu$.

Finally, one finds that the constraint H is also preserved under the duality transformation: $\delta H = \frac{pK}{pK+L} aH$.

In the special case of p–branes in $(p+1)$–dimensions, the Hamiltonian simplifies drastically to

$$H = g^{\alpha\beta}(p_\alpha - \lambda_1 j_\alpha)(p_\beta - \lambda_1 j_\beta) + \lambda_2^{(p+1)L} g^{\alpha\beta} j_\alpha j_\beta \ . \tag{12}$$

One can show that the Hamiltonian constraint and the space reparametrization constraint $H_a = \partial_a x^\mu p_\mu + \partial_a y^\alpha p_\alpha$ are preserved under the full $SL(2)$ duality transformations.

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